vation, in his first memoir on the correction of the lunar elements (*Mem. Astr Soc.*, Vol. XVII), are

$$\delta\lambda = -0''.97 \cos \Omega, \qquad \delta\beta = +2''.17 \cos L.$$

These he has changed to

$$\delta\lambda = -1''.06 \cos \Omega, \qquad \delta\beta = +1''.93 \cos L,$$

in his second memoir (*Mem. Astr. Soc.*, Vol. XXIX).

SYLVESTER'S TERNARY CUBIC FORM.

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Having noticed with interest some of the deductions of Professor Sylvester from what he denominates his Ternary Cubic,* I desire to add a few observations upon this subject and to call attention to some peculiar properties of this cubic which I believe have not been alluded to. The ternary cubic $x^3 - 3xy^2 \pm y^3$ is described by Professor Sylvester† as exhibiting the property, that, if any prime number of the form $18n \pm 1$ or $3(18n \pm 1)$, where n is zero or some integer, can be represented by integer values of x, y in the above expression, then is that number or its nonuple the sum of two cubes. As a matter of fact all such numbers below 537 inclusive (39 in all) were ascertained by Professor Sylvester to be thus representable by integer values of x, y, which he presumed according to the usual canons of induction were sufficient to generalize this remarkable property of the above ternary cubic form to all numbers.‡ Although these actual verifications, are far from establish the generality of the expression for all such numbers; nevertheless, we shall see that this cubic does possess some interesting general properties.

THEOREM I.

If in the equation $x^3 - 3xy^2 \pm y^3 = 18n \pm 1$ or $3(18n \pm 1)$, x and y be any two numbers prime to each other, then will n be zero or some integer.

LEMMA I.

The difference of any two numbers prime to each other is prime to either.

Let x and y represent any two numbers prime to each other; and put x

American Journal of Mathematics, Vol. II. p. 280.

† The same, Vol. II. p. 283.

‡ The same, Vol. III. p. 58.

equal line AB and y equal line CD; then will their difference ED be prime to AB and CD. If ED contains any factor contained an integral number of times in AB, it will also be contained an integral number of times in CE, and would also be contained an integral number of times in CD. Hence AB and CD would have a common factor, which is contrary to the hypothesis; therefore ED is prime to AB and CD. *Q. E. D.*

LEMMA II.

If in the equation $x^3 - 3xy^2 \pm y^3 = 18n \pm 1$ or $3(18n \pm 1)$ there be put for one of the quantities their difference, then will the form of the equation be unchanged and the equation be true.

For x put x, and for y put $x - y$; when

$$x^3 - 3xy^2 + y^3 = 18n \pm 1 \text{ or } 3(18n \pm 1)$$

becomes

$$y^3 - 3yx^2 + x^3 = 18n \pm 1 \text{ or } 3(18n \pm 1).$$

And again put $x - y$ for x and y for y; when

$$x^3 - 3xy^2 - y^3 = 18n \pm 1 \text{ or } 3(18n \pm 1)$$

becomes

$$y^3 - 3yx^2 + x^3 = 18n \pm 1 \text{ or } 3(18n \pm 1).$$

LEMMA III.

If in the equation $x^3 - 3xy^2 \pm y^3 = 18n \pm 1$ or $3(18n \pm 1)$ there be put for one of the quantities their sum, then will the form of the equation be unchanged and the equation be true.

For x put x, and for y put $x + y$; when

$$x^3 - 3xy^2 + y^3 = 18n \pm 1 \text{ or } 3(18n \pm 1)$$

becomes

$$y^3 - 3yx^2 - x^3 = 18n \pm 1 \text{ or } 3(18n \pm 1).$$

And again put $x + y$ for x and y for y; when

$$x^3 - 3xy^2 - y^3 = 18n \pm 1 \text{ or } 3(18n \pm 1)$$

becomes

$$y^3 - 3yx^2 - x^3 = 18n \pm 1 \text{ or } 3(18n \pm 1).$$

LEMMA IV.

By an infinite number of applications of Lemmas II and III all numbers prime to each other are compared in pairs.

Put $x = 1, y = 1$; then by successive applications of Lemma III we shall have

$$\text{System (1)}\left|\begin{array}{lll} x=1 & y=1 & \text{case (1)} \\ x=1 & y=2 & \quad\text{" (2)} \\ x=1 & y=3 & \quad\text{" (3)} \\ x=1 & y=4 & \quad\text{" (4)} \\ x=1 & y=5 & \quad\text{" (5)} \\ \text{etc.} & \text{etc.} & \text{etc.} \\ & \text{to infinity,} & \end{array}\right.$$

and for each case the expression $x^3 - 3xy^2 \pm y^3$ will equal $18n \pm 1$ or $3(18n \pm 1)$. Thus 1 is compared with all the numbers prime to it when system (1) is infinite. To pass to system (2) or that system where 2 is compared with all the numbers prime to it, treat case (2) system (1) by Lemma III, thus:—

$$\text{System (2)}\left|\begin{array}{lll} y=2 & x=1 & \text{case (1)} \\ y=2 & x=3 & \quad\text{" (2)} \\ y=2 & x=5 & \quad\text{" (3)} \\ y=2 & x=7 & \quad\text{" (4)} \\ y=2 & x=9 & \quad\text{" (5)} \\ \text{etc.} & \text{etc.} & \text{etc.,} \end{array}\right.$$

whence 2 is compared with all the numbers prime to it. To obtain system (3), treat case (3) system (1) by composition (Lemma III), thus:—

$$\text{System (3)}\left|\begin{array}{ll} y=3 & x=1 \\ y=3 & x=4 \\ y=3 & x=7 \\ y=3 & x=10 \\ y=3 & x=13 \\ \text{etc.} & \text{etc.} \end{array}\right.$$

To complete this system, evidently we must interpolate another system. Thus: $y = 3, x = 5$; $y = 3, x = 8$; $y = 3, x = 11$; etc., do not appear in the above system. To reach these cases, treat case (2) system (2) by Lemma III, thus:—

$$\text{System (3)}\left|\begin{array}{ll} x=3 & y=5 \\ x=3 & y=8 \\ x=3 & y=11 \\ x=3 & y=14 \\ \text{etc.} & \text{etc.,} \end{array}\right.$$

and system (3) is complete, and 3 is compared with all the numbers prime to it. Similarly treating case (4) system (1) and interpolating from systems (2) and (3), we may obtain system (4) complete. Likewise obtaining system (5) and so on for the other systems, all numbers prime to each other are compared in pairs where the number of systems is infinite. Whence, Lemmas I, II, III, and IV, Theorem I, are true.

We give at random a couple of examples of this theorem. Put $x = 3$, $y = 71$; then

$$3^3 - 3.3.71^2 + 71^3 = 18.17365 - 1.$$

Put $x = 95, y = 4$; then

$$95^3 - 3.95.4^2 + 4^3 = 3\,(18.15794 + 1).$$

THEOREM II.

If in the equation $x^3 - 3xy^2 \pm y^3 = 18n \pm 1$ or $3\,(18n \pm 1)$, where $18n \pm 1$ or $3\,(18n \pm 1)$ represents a prime number, n being zero or some integer and x and y integers, then will x and y be prime to each other.

If they are not prime to each other, assume k equal to their greatest common divisor; so that

$$x = x'k,\ y = y'k,$$

and substitute and divide by k^3; then

$$x'^3 - 3x'y'^2 \pm y'^3 = \frac{18n \pm 1}{k^3} \text{ or } \frac{3\,(18n \pm 1)}{k^3} = \text{integer}$$
$$= 18n' \pm 1 \text{ or } 3\,(18n' \pm 1).$$

But since $18n \pm 1$ or $3\,(18n \pm 1)$ is a prime,* and $\frac{18n \pm 1}{k^3}$ or $\frac{3\,(18n \pm 1)}{k^3}$ is an integer (Theorem I), evidently $k = 1$; whence x and y are prime to each other, their greatest common divisor being unity, which was to be proved.

COR. Multiplying the above ternary cubic by k^3 and subsituting x for $x'k$ and y for $y'k$ we have $x^3 - 3xy^2 \pm y^3 = k^3\,(18n' \pm 1)$ or $3k^3\,(18n' \pm 1)$, where x and y have *G. C. D.* equal k. If $k^3 = 18n'' \pm 1$ evidently

$$x^3 - 3xy^2 \pm y^3 = (18n'' \pm 1)\,(18n' \pm 1) \text{ or } 3\,(18n'' \pm 1)\,(18n' \pm 1)$$
$$= 18n \pm 1 \text{ or } 3\,(18n \pm 1).$$

Therefore, Theorem I may receive this enlargement in the case of composites;

*The possibility of the numbers being divisible by 3 as it may be $3\,(18n \pm 1)$ is the obvious exception to this theorem.

viz.: x and y may have a $G.\ C.\ D.$ which, cubed, equals the canonical expression $18n'' \pm 1$. Evidently k equals the series of numbers $6m \pm 1$, as will be seen from the expansion $k^3 = 18(12m^3 \pm 6m^2 + m) \pm 1$ or $k^3 = 18n'' \pm 1$ where m is zero or some integer.

From Theorem I we may now obtain the general theorem of which Professor Sylvester's expression $x^3 - 3xy^2 \pm y^3$ is a particular case. If the general binary form

$$px^n + qx^{n-1}y + rx^{n-2}y^2 + sx^{n-3}y^3 + tx^{n-4}y^4 + \ldots + wy^n$$

be given and in accordance with Lemma III we put x for x and $x + y$ for y, then evidently

$$\begin{aligned}
px^n &\text{ becomes } px^n, \\
qx^{n-1}y &\text{ becomes } qx^n + qx^{n-1}y, \\
rx^{n-2}y^2 &\text{ becomes } rx^n + 2rx^{n-1}y + rx^{n-2}y^2, \\
&\ldots\ldots\ldots\ldots\ldots \\
vxy^{n-1} &\text{ becomes } vx^n + (n-1)vx^{n-1}y + \frac{(n-1)(n-2)}{2}vx^{n-2}y^2 \\
&\qquad + \frac{(n-1)(n-2)(n-3)}{2.3}vx^{n-3}y^3 + \ldots, \\
wy^n &\text{ becomes } wx^n + nwx^{n-1}y + \frac{n(n-1)}{2}wx^{n-2}y^2 + \ldots + wy^n.
\end{aligned}$$

Impose the condition that the variables in the transformed expression shall simply interchange places, no change taking place in the coefficients, and we shall obtain

$$\begin{aligned}
p + q + r + \ldots + w &= w \\
q + 2r + 3s + \ldots + nw &= v \\
r + 3s + \ldots + \frac{n(n-1)}{2}w &= u \\
s + \ldots + \frac{n(n-1)(n-2)}{2.3}w &= t, \\
\ldots\ldots\ldots\ldots \\
w &= p
\end{aligned}$$

From that system of homogeneous equations determine the values of p, q, r, s, etc., and substitute in the binary form we obtain the general expression for the binary form of any degree enjoying like properties with Professor Sylvester's ternary cubic; in fact that expression, $x^3 - 3\,xy^2 \pm y^3$, is a particular case of the general form. Let it be required, for example, to find this analogue of the fourth degree. Evidently $n = 4$, and

$$\begin{aligned} p+q+r+s+t&=t,\\ q+2r+3s+4t&=s,\\ r+3s+6t&=r,\\ s+4t&=q,\\ t&=p, \end{aligned}$$

whence

$$p=1,\ q=2,\ r=-1,\ s=-2,\ t=1.$$

Substituting these values in the general binary equation, we have

$$x^4+2x^3y-x^2y^2-2xy^3+y^4=24n\pm 1 \text{ or } 4(24n\pm 1).$$

The same general properties, Theorems I and II, may be predicated of this quartic as of the above cubic, inclusive of its enlargement for the composite numbers *G. C. D.* equal $6m\pm 1$. Examples:—

Assume x,y equal any two numbers prime to each other, $x=4$, $y=31$. Evidently

$$4^4+2.4^3.31-4^2.31^2-2.4.31^3+31^4=24.28085+1,\ n=28085.$$

Assume $x=5$, $y=15$. $G.C.D.=5$, $m=1$. Evidently

$$625+3750-5625-33750+50625=24.651+1,\ n=651.$$

SOLUTIONS OF EXERCISES.

2

THREE closely connected tanks, T_1, T_2, T_3, contain Q_1 gallons of water, Q_2 gallons of vinegar, Q_3 gallons of brandy, respectively. A flow is set up from T_1 through T_2 to T_3 and back to T_1 at the rate of 1 gallon per second. The liquids are assumed to mix instantaneously, and the lengths of the connecting pipes are neglected. Show how to calculate the amount of water in each tank at the end of t seconds. [*De Volson Wood.*]

SOLUTION.

Let x_1, x_2, and x_3 be the quantities of water in the three tanks at the time t;